Postcode en Woonplaats: | Jaar van eerste inschrijving: | Datum: 28-01-'04 | Naam docent:

1 $f(x+iy) = x^2 + iy^2 = u(x,y) + i v(x,y)$
waarbij $u(x,y) = x^2$ en $v(x,y) = y^2$
als f holomorf, dan geldt
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ en $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ~~zodat is f holomorf~~

3

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = 2y \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$
dus f is niet holomorf

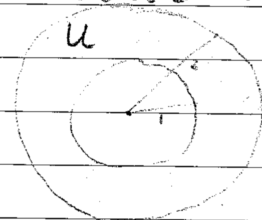
2 $V = \{x+iy \in \mathbb{C} \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$
$\Rightarrow V$ compact
en dus heeft f een maximum M op V,
volgens de stelling van Weierstrass

f is product van holomorfe functies, dus f
holomorf en f continu op V (= closure van V)
Volgens het maximum modulusprincipe ligt
het maximum M van f op de rand van V.
$\Rightarrow |f(z)| < M \quad \forall z \in V \setminus \partial V$, ∂V de rand van V
$M = |f(i)| = 8\sqrt{2}$

2

3 U is open en samenhangend, maar niet
simply connected ... want ... ???

0



dan is $z^\alpha = e^{\alpha \log z}$ *
analytisch op U, dus
holomorf, als $\alpha > 0$
$\Rightarrow f(z) = z^{1/4}$ holomorf op U $\Rightarrow f(z)^4 = (z^{1/4})^4 = z$ op U

* niet gedef op U

4 convergentiestraal r: $\frac{1}{r} = \limsup_{n\to\infty} |a_n|^{1/n}$
en F is gedef. voor $|z| < r$ ($\sum_{n=0} a_n z^n$ convergeert)

$z^2 - 3 = -3z(1-z)$, $\frac{1}{z^2-3} = -\frac{1}{3z}\frac{1}{1-z} = -\frac{1}{3z}\sum_{n=0}^{\infty} z^n$ voor $|z|<1$

2

$\frac{z}{z^2-3} = z\left(-\frac{1}{3z}\sum_{n=0}^{\infty} z^n\right) = -\frac{1}{3}\sum_{n=0}^{\infty} z^n$
$-\frac{3}{z^2-3} = -3\left(-\frac{1}{3z}\sum_{n=0}^{\infty} z^n\right) = \frac{1}{z}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1} = \sum_{n=-1}^{\infty} z^n = \frac{1}{z} + \sum_{n=0}^{\infty} z^n$

$\frac{z-3}{z^2-3} = -\frac{1}{3}\sum_{n=0}^{\infty} z^n + \frac{1}{z} + \sum_{n=0}^{\infty} z^n = \frac{1}{z} + \frac{2}{3}\sum_{n=0}^{\infty} z^n = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{2}{3} z^n$
$a_{-1} = 1$, $a_0 = \frac{2}{3}$, $a_1 = \frac{2}{3}$, $\frac{1}{r} = \limsup |\frac{2}{3}|^{1/n} = 1 \Rightarrow r = 1$

waarbij $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$

holomorf op U omdat er een as weggelaten is

en $f(1) = \text{Log}\, 1 = \log e^{i\cdot 0} = \log e^0 = 0$

$f'(z) = \frac{1}{z}$

$\Rightarrow f(-1) = \text{Log}(-1) = \text{Log}\, e^{i(-\pi)} = \text{Log}\, e^{-i\pi} = -i\pi$

7 neem $f(z) = e^{2\pi i z} - 1$

$e^{2\pi i z} - 1 = 0 \Rightarrow e^{2\pi i z} = 1 \Rightarrow z \in \mathbb{Z}$

nulpunten van f

~~polen~~ binnen C: $z_1 = -1$, $z_2 = 0$, $z_3 = 1$

$$\int_C \frac{1}{e^{2\pi i z} - 1}\, dx = 2\pi i \left(-\text{Res}_{z_1}\left(\tfrac{1}{f}\right) - \text{Res}_{z_2}\left(\tfrac{1}{f}\right) - \text{Res}_{z_3}\left(\tfrac{1}{f}\right)\right)$$

$f'(z) = 2\pi i e^{2\pi i z} (-1)$

$f'(z_1) = f'(z_2) = f'(z_3) = 2\pi i \cdot 1 - 1 = 2\pi i - 1 \neq 0$

$\text{Res}_{z_i}\left(\frac{1}{f}\right) = \frac{1}{2\pi i - 1}$ $\quad \frac{1}{2\pi i}$

$$\int_C \frac{1}{f}\, dz = 2\pi i \left(-3 \frac{1}{2\pi i - 1}\right) = -\frac{6\pi i}{2\pi i - 1}$$

8 $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \rightarrow e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}$

$z e^{\frac{1}{z}} = z \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!} = \sum_{n=-\infty}^{0} \frac{z^{n+1}}{(-n)!} = \sum_{n=-\infty}^{1} \frac{z^n}{(-(n-1))!} = z + \sum_{n=-\infty}^{0} \frac{z^n}{(-(n-1))!}$

$(z+1) e^{1/z} = z e^{1/z} + e^{1/z} = \sum_{n=-\infty}^{0} \frac{z^n}{(-(n-1))!} + z + \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}$

$= z + \sum_{n=-\infty}^{0} \left(\frac{1}{(-(n-1))!} + \frac{1}{(-n)!}\right) z^n = f(z)$

f heeft essentiële singulariteit in 0

(f heeft z^{-n} voor oneindig veel n'en $\in \mathbb{N}$)

9 $z^2 + n^5 = 0 \Rightarrow z^2 = -n^5 \Rightarrow z_1 = i\sqrt{n^5}$, $z_2 = -i\sqrt{n^5}$

~~polen~~

$f = z^2 + n^5$ $\quad f' = 2z$ $\quad f'(z_1) \neq 0$ en $f'(z_2) \neq 0$

↳ simpele polen van $\frac{1}{f}$

$\Rightarrow$ orde 1

$\frac{1}{f} = \frac{1}{(z - z_1)(z - z_2)}$

$\text{Res}_{z_1} \frac{1}{f} = \frac{1}{z_1 - z_2} = \frac{1}{2i\sqrt{n^5}}$

$\text{Res}_{z_2} \frac{1}{f} = \frac{1}{z_2 - z_1} = -\frac{1}{2i\sqrt{n^5}} = \frac{i}{2\sqrt{n^5}}$

alle singulariteiten zijn polen $\Rightarrow$ meromorf reeles is

Afdeling Wiskunde en Informatica R.U.G.

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10 9 $f(z) = z^4 + 2 \qquad f'(z) = 4z^3 + 2$

$f(z) = z^4 + 2 = 0 \Rightarrow z^4 = -2 \Rightarrow z = 2^{1/4} e^{k\pi i/4} \qquad k = -1, 1, -3, 3$

in bovenhalfvlak hebben we $z_1 = 2^{1/4} e^{\pi i/4}$ en $z_2 = 2^{1/4} e^{3\pi i/4}$

$f'(z_1) = 4 \cdot 2^{3/4} e^{3\pi i/4} + 2 = 2^2 \cdot 2^{3/4} \cdot \frac{1}{2}\sqrt{2}(i-1) + 2$

$= 2^{9/4}(i-1) + 2 \quad \cancel{\quad} \neq 0$

$f'(z_2) = 4 \cdot 2^{3/4} e^{9\pi i/4} + 2 = 2^{9/4}(i+1) + 2 \neq 0$

$g(z) = z^2 + 1$

$g(z_1) = 2^{1/4} e^{2\pi i/4} + 1 = 2^{1/4} i + 1 \neq 0$

$g(z_2) = 2^{1/4} e^{6\pi i/4} + 1 = -2^{1/4} i + 1 \neq 0$

WAAROM

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+2}\,dx = 2\pi i \left(Res_{z_1}\left(\frac{g(z)}{f(z)}\right) + Res_{z_2}\left(\frac{g(z)}{f(z)}\right) \right)$$

$$= 2\pi i \left(\frac{2^{1/4} i + 1}{2^{9/4}(i-1)+2} + \frac{-2^{1/4} i + 1}{2^{9/4}(i+1)+2} \right) = \ldots$$

6 $\cancel{a_n = \int \frac{f(z)}{z^n}\,dz \leq \frac{|f(z)|}{z^n} \leq 2^{1-n}}$

$$|a_n| = \left| \frac{f^{(n)}(0)}{n!} \right| = \left| \frac{1}{n!} \frac{1}{2\pi i} \int_{C_r} \frac{f^{(n)}(z)}{z}\,dz \right|$$

$$= \left| \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z^{n+1}}\,dz \right|$$

$$\cancel{\leq \frac{1}{2\pi i} \int_{C_r} \left| \frac{f(z)}{z^{n+1}} \right| dz \leq \frac{1}{2\pi i} L(C_R) \|f\|}$$

$$\cancel{\leq \frac{1}{2\pi i} \int_{C_r} \left| \frac{2}{z^{n+1}} \right| dz}$$

$$\cancel{\leq \frac{1}{\pi i} \frac{1}{n} \int_{C_r} \left[\frac{1}{z^n} \right]}$$

$$\leq \left| \frac{1}{2\pi i} L(C_R) \right| \left\| \frac{f(z)}{z^{n+1}} \right\|_{C_R}$$

supnorm

$$\leq \frac{2}{2^{n+1}} = \frac{1}{2^n} = 2^{-n} \leq 2^{1-n}$$

vervolg 2

berekien M :

bekijk maximum van $|f(z)|$ op $x=0 \quad 0 \leq y \leq 1$

$x=1 \quad 0 \leq y \leq 1$

$y=0 \quad 0 \leq x \leq 1$

$y=1 \quad 0 \leq x \leq 1$

en vergelijk $|f(z)|$, neem de hoogste